

On the importance of nonlinear couplings in large-scale neutrino streams

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Abstract. We propose a procedure to evaluate the impact of nonlinear couplings on the evolution of massive neutrino streams in the context of large-scale structure growth. Such streams can be described by general nonlinear conservation equations, derived from a multiple-flow perspective, which generalize the conservation equations of non-relativistic pressureless fluids. The relevance of the nonlinear couplings is quantified with the help of the eikonal approximation applied to the subhorizon limit of this system. It highlights the role played by the relative displacements of different cosmic streams and it specifies, for each flow, the spatial scales at which the growth of structure is affected by nonlinear couplings. We found that, at redshift zero, such couplings can be significant for wavenumbers as small as $k = 0.2 h/\text{Mpc}$ for most of the neutrino streams.

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1 Introduction

Statistical properties of the large-scale structure of the universe have long been proposed as an efficient instrument to constrain cosmological parameters. In this context, a careful account of the role played by massive neutrinos is crucial. So far, it has often been overlooked in the nonlinear or quasilinear regime because of the technical complexity specific to the study of massive neutrinos (see recent attempts in [1–3]). This is all the more unfortunate that cosmological observations can fruitfully improve our knowledge of those particles. The signature of neutrino masses on cosmological observables is indeed expected to be significant enough for those masses to be constrained observationally [4–9].

In the linear regime, the effect of neutrinos is now well understood (see refs. [10–12]). The need for nonlinear corrections in their equations of motion has been raised because the cosmological observations that are the most sensitive to neutrinos masses, i.e. for wavenumbers in the 0.1-0.2 h/Mpc range, precisely correspond to the mildly nonlinear regime. To deal with this issue, several strategies can be adopted. Until recently, in analytic works, the neutrino fluids had always been treated in the linear regime, nonlinear couplings being introduced in the dark matter description only. This nonlinear treatment can be implemented with the help of the Renormalization Group time-flow approach [1, 13]. Improvements upon such schemes have been proposed in [3], which consists in a hybrid approach that matches the full Boltzmann hierarchy to an effective two-fluid description at an intermediate redshift. [14] is, for its part, a systematic perturbative expansion of the Vlasov equation in which high-order corrections to the neutrino density contrast are computed without the explicit need to track the perturbed neutrino momentum distribution.

Ideally, however, the fully nonlinear evolution of the neutrino fluid should be depicted. A natural way to do so would be to take inspiration of the standard linear description, which relies on the Boltzmann equation, and extend the harmonic decomposition of the phase-space distribution function to the nonlinear regime. This has been done in [15] but this method turned out to be particularly difficult to handle. In [16, 17], we proposed to describe massive neutrinos as a superposition of single-flow fluids, the equations of motion of each of them being written in the nonlinear regime.

In this paper, we are interested in exploiting those theoretical developments in order to identify the scales at which nonlinear couplings in the neutrino fluids are expected to

play a significant role. In order to do so, we apply the eikonal approximation to the nonlinear equations of motion. Note that this approximation had already been exploited in the literature to develop a Perturbation Theory approach for cold dark matter ([18]) and had proved to be able to capture the leading coupling effects.

The article is structured as follows. In section 2, we recall the form of the nonlinear equations of motion describing the time evolution of non-interacting fluids, relativistic or not, when using a multi-fluid approach. Section 3 explains in detail how the eikonal approximation can be implemented in those equations and emphasizes the key role of relative displacement fields. Finally, in section 4, power spectra of the relative displacements between neutrino fluids and the cold dark matter component are presented. The impact on the growth of large-scale structure is then discussed in a quantitative way.

2 Nonlinear equations of motion (multi-fluid description)

Following [16, 17], it is now clear that any non-interacting relativistic fluid can be divided into several flows, each of them evolving then independently until first shell-crossings. In cosmology, this approach obviously applies to massive neutrinos since they are free-streaming. In this framework, each flow can be defined as the collection of all the particles (with a mass m) having the same comoving momentum. They are entirely characterized by two coupled fields, namely the comoving number density n_c and momentum P_μ . Those fields obey the following general equations,

$$\frac{\partial}{\partial \eta} n_c + \frac{\partial}{\partial x^i} \left(\frac{P^i}{P^0} n_c \right) = 0, \quad (2.1)$$

with $P^\mu(\eta, x^i) = g^{\mu\nu} P_\nu(\eta, x^i)$ and $P^\mu P_\mu = -m^2$, $g_{\mu\nu}$ being the metric, and

$$P^\nu P_{\mu,\nu} = \frac{1}{2} P^\sigma P^\nu g_{\sigma\nu,\mu}. \quad (2.2)$$

These equations directly ensue from the matter and momentum conservation equations, applied to each flow. At this stage, no perturbative expansion of the metric is involved. The properties of the whole fluid are then inferred by examining an appropriate number of such flows, each of them being labeled by the initial value of its field P_i , denoted τ_i (found to be constant at zeroth order in Perturbation Theory). The initial number density of particles in each flow is constrained by the choice of initial conditions. For instance, the case of adiabatic initial conditions is described in detail in [16].

We are interested here in equations i) involving linearized metric perturbations (but non-linearized fields) and ii) rid of the coupling terms that are subdominant at subhorizon scales. They can be written in terms of the number density contrast

$$\delta_{\tau_i}(\eta, x^i) = \frac{n_c(\eta, x^i; \tau_i)}{n_c^{(0)}(\tau_i)} - 1 \quad (2.3)$$

and of the velocity divergence field (in units of $-\mathcal{H}$, \mathcal{H} being the conformal Hubble constant)

$$\theta_{\tau_i}(\eta, x^i) = -\frac{\partial_i P_i(\eta, x^i; \tau_i)}{ma\mathcal{H}} \quad (2.4)$$

since the field P_i has found to be potential in this regime¹. For a generic perturbed Friedmann-Lemaître metric², whose time variable is the conformal time η ,

$$ds^2 = a^2(\eta) \left[-(1 + 2A)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + h_{ij})dx^i dx^j \right], \quad (2.5)$$

one has in Fourier space for the mode characterized by the wave vector \mathbf{k} ([17])

$$\begin{aligned} \left(a\partial_a - i\frac{\mu k\tau}{\mathcal{H}\tau_0} \right) \delta_{\tau_i}(\mathbf{k}) + \frac{ma}{\tau_0} \left(1 - \frac{\mu^2\tau^2}{\tau_0^2} \right) \theta_{\tau_i}(\mathbf{k}) = \\ -\frac{ma}{\tau_0} \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \alpha_R(\mathbf{k}_1, \mathbf{k}_2; \tau_i) \delta_{\tau_i}(\mathbf{k}_1) \theta_{\tau_i}(\mathbf{k}_2), \quad (2.6) \\ \left(1 + a\frac{\partial_a \mathcal{H}}{\mathcal{H}} + a\partial_a - i\frac{\mu k\tau}{\mathcal{H}\tau_0} \right) \theta_{\tau_i}(\mathbf{k}) - \frac{k^2}{ma\mathcal{H}^2} \mathcal{S}_{\tau_i}(\mathbf{k}) = \\ -\frac{ma}{\tau_0} \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \beta_R(\mathbf{k}_1, \mathbf{k}_2; \tau_i) \theta_{\tau_i}(\mathbf{k}_1) \theta_{\tau_i}(\mathbf{k}_2). \quad (2.7) \end{aligned}$$

We have introduced here

$$\mu = \frac{k_i \tau_i}{k\tau}, \tau^2 = \tau_i \tau_i \text{ and } \tau_0 = -\sqrt{m^2 a^2 + \tau_i^2}. \quad (2.8)$$

Besides, $\mathcal{S}_{\tau_i}(\mathbf{k})$ is a source term given by

$$\mathcal{S}_{\tau_i}(\mathbf{k}) = \tau_0 A(\mathbf{k}) + \vec{\tau} \cdot \vec{B}(\mathbf{k}) - \frac{1}{2} \frac{\tau_i \tau_j}{\tau_0} h_{ij}(\mathbf{k}). \quad (2.9)$$

These equations contain also the generalized kernel functions, adapted to relativistic flows,

$$\alpha_R(\mathbf{k}_1, \mathbf{k}_2; \tau) = \delta_{\text{Dirac}}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \frac{(\mathbf{k}_1 + \mathbf{k}_2)}{k_2^2} \cdot \left[\mathbf{k}_2 - \vec{\tau} \frac{\mathbf{k}_2 \cdot \vec{\tau}}{\tau_0^2} \right], \quad (2.10)$$

$$\beta_R(\mathbf{k}_1, \mathbf{k}_2; \tau) = \delta_{\text{Dirac}}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \frac{(\mathbf{k}_1 + \mathbf{k}_2)^2}{2k_1^2 k_2^2} \left[\mathbf{k}_1 \cdot \mathbf{k}_2 - \frac{\mathbf{k}_1 \cdot \vec{\tau} \mathbf{k}_2 \cdot \vec{\tau}}{\tau_0^2} \right]. \quad (2.11)$$

As mentioned in [17], they are extensions of the kernel functions found for pressureless fluids of non-relativistic species (see [19] for details in this context.).

In any practical implementation, it is necessary to consider a collection of N streams. In that case, the general equation of motion can conveniently be written in terms of the time-dependent $2N$ -uplet,

$$\Psi_a(\mathbf{k}) = (\delta_{\tau_1}(\mathbf{k}), \theta_{\tau_1}(\mathbf{k}), \dots, \delta_{\tau_N}(\mathbf{k}), \theta_{\tau_N}(\mathbf{k}))^T. \quad (2.12)$$

Before shell-crossing, it can incorporate all the relevant species (neutrinos, dark matter, baryons) as long as they interact only via gravitation. In this context, the equations (2.6) and (2.7) of all the flows can formally be recast in the form³

$$\partial_z \Psi_a(\mathbf{k}) + \Omega_a^b \Psi_b(\mathbf{k}) = \gamma_a^{bc}(\mathbf{k}_1, \mathbf{k}_2) \Psi_b(\mathbf{k}_1) \Psi_c(\mathbf{k}_2), \quad (2.13)$$

¹This property, rigorously demonstrated in [17], generalizes that of non-relativistic species.

²Units are chosen so that the speed of light in vacuum is equal to unity.

³The Einstein notation for the summation over repeated indices is adopted and, in the right hand side of this equation, it is assumed that the wave modes are integrated over.

where $\partial/\partial z \equiv a\partial/\partial a$ and the indices a and b run from 1 to $2N$. The matrix elements Ω_a^b gather the linear couplings. They contain in particular the way in which the source terms $\mathcal{S}_{\tau_i}(\mathbf{k})$ can be re-expressed as a function of the $2N$ -uplet elements. The left hand side of this equation is nothing but the linear field evolution. The right hand side contains the coupling terms. More precisely, the *symmetrized vertex* matrix $\gamma_a^{bc}(\mathbf{k}_1, \mathbf{k}_2)$ describes the nonlinear interactions between the Fourier modes. It is given by

$$\gamma_{2p-1}^{2p-1, 2p}(\mathbf{k}_1, \mathbf{k}_2) = -\frac{ma}{2(\tau_p)_0} \alpha_R(\mathbf{k}_1, \mathbf{k}_2, \tau_p), \quad (2.14)$$

$$\gamma_{2p}^{2p, 2p}(\mathbf{k}_1, \mathbf{k}_2) = -\frac{ma}{(\tau_p)_0} \beta_R(\mathbf{k}_1, \mathbf{k}_2, \tau_p), \quad (2.15)$$

with $\gamma_a^{bc}(\mathbf{k}_1, \mathbf{k}_2) = \gamma_a^{cb}(\mathbf{k}_2, \mathbf{k}_1)$ and $\gamma_a^{bc} = 0$ otherwise.

3 The eikonal approximation

The eikonal approximation⁴, developed in [18], is based on the observation that the amplitudes of the kernel functions describing mode couplings, α_R and β_R , significantly depend on the ratio between the wave numbers at play. This has been observed for non-relativistic fluids and we show here that it is the case for relativistic fluids too. This property leads to the idea that the right hand side of eq. (2.13) can be split into two integration domains. One is called the hard domain and encompasses modes whose wavelengths are of the same order (and for which the coupling functions are always finite). The other one is referred to as the soft domain. It is made of modes of very different wavelengths for which the coupling functions, of the order of the wavelength ratio, are large.

The main idea is that there are regimes in which the dominant coupling structure is in the soft domain. This is the case for instance in early-time fluids containing baryons and cold dark matter. At the time of recombination, the baryon velocity drops steeply whereas the velocity of cold dark matter is not affected by decoupling. It means that, at intermediate scales (i.e. between the Silk damping length and the sound horizon), the relative velocity v_{rel} between the baryon flow and the one of cold dark matter is substantial. Because of this, for scales at which $k > aH/\langle v_{\text{rel}}^2 \rangle^{1/2}$ (at decoupling), the wavelength of gravitational potential wells is too small for baryons to fall in them before being pushed towards another direction. Eventually, this phenomenon induces a damping of the matter power spectrum. The relative motion between cold dark matter and baryons and its effects on the matter power spectrum have been highlighted in [21, 22]. Such studies illustrate the relevance of couplings between large scale modes and small scale modes in the framework of structure formation. For instance, in the case of mixtures of baryons and cold dark matter, the typical coherence length of the relative velocity field is of the order of few Mpc, which is much larger than the scales at which basic baryonic objects start to form under gravitational clustering (of the order of 10 kpc). The same formalism can be used to obtain the large k behavior of the propagators in case of a single pressureless fluid, reproducing the results obtained in [23, 24]. In the same spirit, we propose here to investigate the impact of the relative motion between given neutrino streams and the cold dark matter fluid with the help the eikonal approximation. It will allow us to infer the amplitude of neutrino coupling effects.

⁴In this context, the term refers to diagram resummations performed in quantum electrodynamic field equations, [20].

For convenience, let us assume that the soft domain is obtained for $k_2 \ll k_1$ in eq. (2.13). We have then $\mathbf{k} = \mathbf{k}_1$ and the contribution corresponding to the soft domain can be viewed as a mere corrective term in the *linear* equation describing the evolution of the mode \mathbf{k} . In other words, eq. (2.13) can be rewritten

$$\begin{aligned} \frac{\partial}{\partial z} \Psi_a(z, \mathbf{k}) + \Omega_a^b(z, \mathbf{k}) \Psi_b(z, \mathbf{k}) - \Xi_a^b(z, \mathbf{k}) \Psi_b(z, \mathbf{k}) \\ = \left[\gamma_a^{bc}(\mathbf{k}_1, \mathbf{k}_2) \Psi_b(\mathbf{k}_1, z) \Psi_c(\mathbf{k}_2, z) \right]_{\mathcal{H}}, \end{aligned} \quad (3.1)$$

with

$$\Xi_a^b(\mathbf{k}, z) \equiv 2 \int_S d^3 \mathbf{q} \text{ eik. } \gamma_a^{bc}(\mathbf{k}, \mathbf{q}) \Psi_c(\mathbf{q}, z). \quad (3.2)$$

The soft momenta q (i.e. $q \ll k$) at play in eq. (3.2) are integrated over so that $\Xi_a^b(z, \mathbf{k})$ is independent on $\Psi_a(z, k)$. It is a mere time and scale dependent matrix. The fact that the integration domain is restricted to the soft wave numbers in eq. (3.2) is the key element. Conversely, in the right-hand side of eq. (3.1), the implicit convolution product excludes the soft domain (i.e. all the modes concerned have comparable wavelengths). When the contribution of the hard domain is negligible, eq. (3.1) can then be viewed as the equation of motion of the mode \mathbf{k} evolving in a medium perturbed by large-scale modes. It therefore encodes the way in which long-wave modes alter the growth of structure. Once $\Xi_a^b(z, \mathbf{k})$ is given, eq. (3.1) can be solved as a linear equation. This is precisely the eikonal approximation of the global equation of motion.

In practice, applying the eikonal approximation to $\Xi_a^b(\mathbf{k}, z)$ means that the vertex values that appear in this quantity have to be computed assuming $k_2 \ll k_1$. In this framework, one deduces from eqs. (2.10), (2.11), (2.14) and (2.15) that the eikonal limit of the vertex elements is

$$\text{eik. } \gamma_{2p}^{bc}(\mathbf{k}, \mathbf{k}_2) = -\delta_{2p}^b \delta_{2p}^c \frac{ma}{2k_2^2 (\tau_p)_0} \mathbf{k} \cdot \left(\mathbf{k}_2 - \frac{\mathbf{k}_2 \cdot \vec{\tau}_p}{(\tau_p)_0^2} \vec{\tau}_p \right), \quad (3.3)$$

$$\text{eik. } \gamma_{2p-1}^{bc}(\mathbf{k}, \mathbf{k}_2) = -\delta_{2p-1}^b \delta_{2p}^c \frac{ma}{2k_2^2 (\tau_p)_0} \mathbf{k} \cdot \left(\mathbf{k}_2 - \frac{\mathbf{k}_2 \cdot \vec{\tau}_p}{(\tau_p)_0^2} \vec{\tau}_p \right). \quad (3.4)$$

This expression depends on each flow through its initial momentum $\vec{\tau}_p$, which vanishes in the standard non-relativistic equations. Besides, $(\tau_p)_0 \rightarrow -ma$ in the non-relativistic limit so that one recovers the expected formulae in this limit (see [21, 25]).

In the following, we exploit the consequences of this approximation in order to evaluate the relevance of the coupling terms. As a first step, we can notice from eqs. (3.2), (3.3) and (3.4) that the two non-zero elements coming from the flow labeled by τ_p in the Ξ matrix, Ξ_{2p}^{2p} and Ξ_{2p-1}^{2p-1} , are proportional to the velocity divergence of large-scale modes. Thus we can write

$$\int_{z_0}^z \Xi_a^b(z', \mathbf{k}) dz' = i \mathbf{k} \cdot \mathbf{d}_p(z_0, z) \delta_a^b, \quad (3.5)$$

where a and b are either $2p$ or $2p-1$ and where d_p is the total displacement field induced by the large-scale modes in the fluid labeled by τ_p . It reads necessarily

$$\mathbf{d}_p(z, z_0) = i \int_{z_0}^z dz' \int d^3 \mathbf{q} \frac{ma}{q^2 (\tau_p)_0} \left(\mathbf{q} - \frac{\mathbf{q} \cdot \vec{\tau}_p}{(\tau_p)_0^2} \vec{\tau}_p \right) \Psi_{2p}(z', \mathbf{q}). \quad (3.6)$$

Note that this displacement is superimposed on the zeroth order displacement field induced by the homogeneous momentum of the flow. This background displacement field is given by (see eqs. (2.6) and (2.7))

$$\mathbf{d}_p^{(0)} = - \int_{z_0}^z \frac{\vec{\tau}_p}{\mathcal{H}(\tau_p)_0}. \quad (3.7)$$

We will discuss this point in greater detail in the next section.

The impact of the eikonal correction introduced in the equation of motion depends on the way in which the various large-scale modes contribute to the displacement fields \mathbf{d}_p . The nature of the displacement is crucial in this context. In particular, one expects global displacement fields, which affect all species in a similar way, to induce a mere phase shift in the solution of the eikonal limit of (3.1). Remarkably, such displacements have no impact on power spectra, provided that the fields at play are evaluated at the same time (see the end of this section). Using the standard language of cosmology, we call those particular displacements “adiabatic displacements” in the following. On the contrary, relative displacements between species can induce a damping in power spectra. It is simply due to the fact that species must evolve in phase (at least during a small period of time) for couplings between them to generate a significant growth of perturbations. This phenomenon has been highlighted for the first time in [21] and reconsidered in [22]. In this paper, we show that such considerations can be extended to the study of relativistic fluids.

Knowing this, it is convenient to decompose displacement fields into

$$d_p(z, z_0) = d_{\text{adiab}}(z, z_0) + \delta d_p(z, z_0), \quad (3.8)$$

where d_{adiab} denotes naturally the adiabatic part. The other part, δd_p , represents relative motions between fluids. For non-relativistic species, it is known that the most growing mode is part of the adiabatic modes. Since neutrinos become non-relativistic at late time, one expects the most growing mode of each fluid of neutrinos to become also an adiabatic mode ultimately.

When considering only the adiabatic part of the displacement field, the solution of the eikonal equation of motion is easy to find. Its form is related to the extended Galilean invariance of the equations of motion, well established for non-relativistic species and uncovered for the system (2.6)-(2.7) in [17]. Indeed, it has been shown that this system is invariant under the following transformations,

$$\tilde{x}^i = x^i + d_i(z), \quad (3.9)$$

$$\tilde{z} = z, \quad (3.10)$$

$$\tilde{\psi}_a(\tilde{z}, \tilde{\mathbf{x}}) = \psi_a(\eta, \mathbf{x}), \quad (3.11)$$

where the last transformation can equivalently be written $\tilde{\psi}_a(\tilde{\eta}, \mathbf{k}) = \exp(i\mathbf{k} \cdot \mathbf{d}(\eta))\psi_a(\eta, \mathbf{k})$. It means that a homogeneous time-dependent displacement which disrupts the medium can be re-absorbed in a global phase shift of the Fourier transforms of the fields. Here, we are interested in making the large-scale adiabatic displacement fields play the role of the disturbers of the medium.

More explicitly, it is known that the solution of the standard linear system can be fully described with the help of its Green function, $g_a^b(z, z_0; \mathbf{k})$, defined in such a way that (see appendix A for details)

$$\Psi_a(z, \mathbf{k}) = g_a^b(z, z_0; \mathbf{k})\Psi_b(z_0, \mathbf{k}), \quad (3.12)$$

with z and z_0 two arbitrary times. Besides, when the displacement field is purely adiabatic, the Green functions ξ_a^b of the corrected linear system (i.e. the linear system in which an eikonal correction has been added) are related to those of the naked theory by

$$\xi_a^b(z, z_0; \mathbf{k}) = g_a^b(z, z_0; \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{d}_{\text{adiab}}(z, z_0)). \quad (3.13)$$

A direct consequence of the symmetry property is that, after a transformation of the form (3.11), ensemble averages of any product of fields $\psi_a(z_a, \mathbf{k}_a)$ become proportional to $\exp[i \sum_a \mathbf{k}_a \cdot \mathbf{d}_{\text{adiab}}(z_a)]$, which is unity when the fields are computed at equal time. Indeed, statistical homogeneity imposes $\sum_a \mathbf{k}_a = 0$. This is the reason why equal time spectra or poly-spectra are not sensitive to the presence of adiabatic displacements.

Yet, we are interested in all contributions to the displacement field, including those that induce large motions between species. As already mentioned, in early-time mixtures of baryons and cold dark matter, those relative displacements (i.e. those non-adiabatic contributions that develop in the nonlinear regime) are the leading contributions to the nonlinear evolution of the power spectrum. To sketch the impact of massive neutrinos on the nonlinear growth rate, we evaluate in the next section the amplitude of the relative displacements involving them and we compute the corresponding power spectra.

4 Relative displacements and power spectra: quantitative results

As stressed in the previous section, in the eikonal approximation, the displacement fields that can cause a damping of the growth of structure are those that differ from one fluid to another. Hence, in this section, we compute on the one hand the power spectrum of the total displacement field of cold dark matter and, on the other hand, the power spectra of the relative displacement fields of relativistic flows (with respect to the a priori dominant cold dark matter component). Note that, in perturbation theory, such displacements are sums of terms of different orders (the zeroth order contribution $\mathbf{d}_p^{(0)}$ being defined in eq. (3.7)),

$$\mathbf{d}_p(z, z_0) = \mathbf{d}_p^{(0)}(z, z_0) + \mathbf{d}_p^{(1)}(z, z_0) + \dots \quad (4.1)$$

Note also that the damping due to non-homogeneous corrections can be significant only if the considered flow is non-relativistic (since in that case $\mathbf{d}_\tau^{(0)}(z, z_0)$ is small) or if \mathbf{k} is orthogonal to the zeroth order contribution (since in that case $\mathbf{k} \cdot \mathbf{d}_\tau^{(0)}(z, z_0)$ is small compared with the other contributions). Hence, in the following, we compute the expected values of $\mathbf{k} \cdot (\mathbf{d}_p - \mathbf{d}_{\text{cdm}})$ as a function of the angle between the initial momentum $\vec{\tau}$ of the considered fluid and \mathbf{k} . Calculations are performed in the conformal Newtonian gauge,

$$ds^2 = a^2(\eta) \left[- (1 + 2\psi) d\eta^2 + (1 - 2\phi) dx^i dx^j \delta_{ij} \right]. \quad (4.2)$$

This choice will allow us to take advantage of the numerical work presented in [16]. First, let us define two transfer functions D_{cdm} and $D_p^{(\alpha)}$ as

$$\int_{z_0}^z dz' \frac{ma}{-(\tau_p)_0} \left(\frac{\tau_p}{(\tau_p)_0} \right)^\alpha \theta_{\tau_p}(z', \mathbf{q}) = D_p^{(\alpha)}(z, z_0, \mathbf{q}) \psi_{\text{init}}(\mathbf{q}), \quad (4.3)$$

$$\int_{z_0}^z dz' \theta_{\text{cdm}}(z', \mathbf{q}) = D_{\text{cdm}}(z, z_0, \mathbf{q}) \psi_{\text{init}}(\mathbf{q}), \quad (4.4)$$

where ψ_{init} is the initial value of the potential ψ . The statistical properties of those quantities are entirely encoded in their initial power spectra $P_\psi(q)$, defined so that

$$\langle \psi_{\text{init}}(\mathbf{q}) \psi_{\text{init}}(\mathbf{q}') \rangle = (2\pi)^3 \delta_{\text{Dirac}}(\mathbf{q} + \mathbf{q}') P_\psi(q). \quad (4.5)$$

Using the expressions of the transfer functions, the contribution of each mode to the displacement field of each relativistic flow reads

$$\mathbf{d}_p(z, z_0; \mathbf{q}) = -i \left(\frac{\mathbf{q}}{q^2} D_p^{(0)}(z, z_0, \mathbf{q}) - \frac{\vec{\tau}_p \cdot \mathbf{q}}{q^2 (\tau_p)^2} \vec{\tau}_p D_p^{(2)}(z, z_0, \mathbf{q}) \right) \psi_{\text{init}}. \quad (4.6)$$

Besides, for the cold dark matter component, one simply has

$$\mathbf{d}_{\text{cdm}}(z, z_0; \mathbf{q}) = -i \frac{\mathbf{q}}{q^2} D_{\text{cdm}}(z, z_0, \mathbf{q}) \psi_{\text{init}}(q). \quad (4.7)$$

Furthermore, one can notice that the displacement field of a relativistic flow along an arbitrary direction \mathbf{k} depends on the angles between both \mathbf{q} and $\vec{\tau}$ and \mathbf{k} and $\vec{\tau}$. After integration over the other angles, one finds for the variances of respectively $\mathbf{k} \cdot \mathbf{d}_{\text{cdm}}$ and $\mathbf{k} \cdot (\mathbf{d}_p - \mathbf{d}_{\text{cdm}})$,

$$\langle (\mathbf{k} \cdot \mathbf{d}_{\text{cdm}})^2 \rangle = 4\pi k^2 \int dq P_\psi(q) \frac{1}{3} |D_{\text{cdm}}(\mathbf{q})|^2 \quad (4.8)$$

and

$$\langle (\mathbf{k} \cdot (\mathbf{d}_p - \mathbf{d}_{\text{cdm}}))^2 \rangle = 2\pi k^2 \times \int dq P_\psi(q) \int_{-1}^1 d\mu \left[\frac{1}{2} (1 - \mu_k^2)(1 - \mu^2) |D_p^{(0)}(\mathbf{q}) - D_{\text{cdm}}(\mathbf{q}) - D_p^{(2)}(\mathbf{q})|^2 + \mu^2 \mu_k^2 |D_p^{(2)}(\mathbf{q})|^2 \right], \quad (4.9)$$

where μ_k is the Cosine of the angle between the initial momentum of the flow $\vec{\tau}_p$ and \mathbf{k} and where an integration is made over μ , Cosine of the angle between $\vec{\tau}_p$ and \mathbf{q} . In eq. (4.9), one can notice that the dependence of the r.m.s. with respect to μ_k is such that it does not vanish either for an initial momentum $\vec{\tau}_p$ orthogonal to \mathbf{k} (i.e. when $\mu_k = 0$) or for an initial flow momentum along \mathbf{k} (i.e. when $\mu_k = 1$).

On Figure 1, we present the per mode contribution to the right hand side of eq. (4.9) in the particular case of neutrino fluids for $\mu_k = 0$ on the left panel and $\mu_k = 1$ on the right panel. The results have been computed assuming a single species of neutrinos whose mass is $m_\nu = 0.3$ eV and using the cosmological parameters derived from the Five-Year Wilkinson Microwave Anisotropy Probe (WMAP 5) observations. Besides, the values of the initial power spectra have been obtained under the assumption that, in the cosmological model we adopt, metric fluctuations are initially adiabatic and characterized by the scalar spectral index $n_s \approx 0.96$. We can see that the resulting neutrino power spectra are comparable in amplitude to the cold dark matter one (represented by a thick dashed line).

Denoting $\sigma_{d_{\text{cdm}}} = 1/k \langle (\mathbf{k} \cdot \mathbf{d}_{\text{cdm}})^2 \rangle^{1/2}$ and $\sigma_{d_\tau} = 1/k \langle (\mathbf{k} \cdot (\mathbf{d}_p - \mathbf{d}_{\text{cdm}}))^2 \rangle^{1/2}$ we find that σ_{d_τ} is of the order of 2 to about $5h^{-1}\text{Mpc}$ for the cosmological modes we used for which $\sigma_{d_{\text{cdm}}} \approx 6h^{-1}\text{Mpc}$. This result depends of course on the flow considered and is actually of the order of the cold dark matter value. What does it mean? Similarly to what happens in mixtures of baryons and cold dark matter, one expects the perturbation growth to be damped for wave numbers larger than or comparable to $1/\sigma_{d_\tau}$, for which $\mathbf{k} \cdot \mathbf{d}_\tau^{(1)}$ is expected to be finite. Besides, such a non-adiabatic damping is potentially larger than the damping due to the homogeneous displacements $\mathbf{d}_p^{(0)}$ of each flow when μ_k is close to zero. A precise determination of the amplitude of these effects would require a full analysis of the nonlinear evolution of the system. We leave this for a future study.

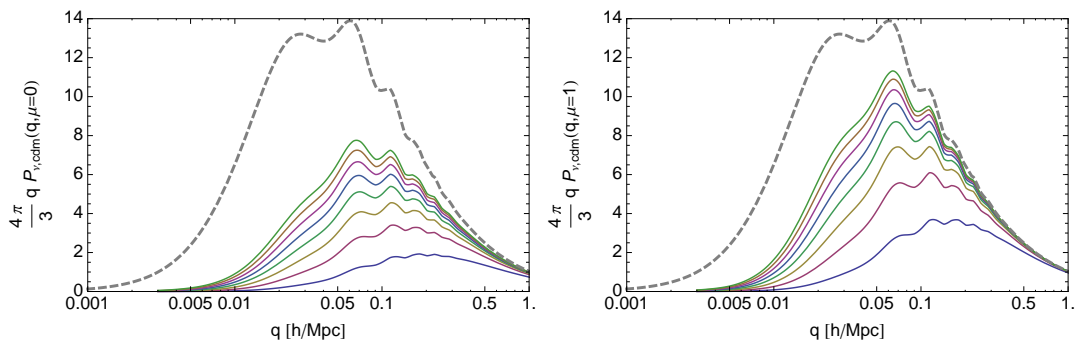


Figure 1. Power spectrum of the relative displacement as a function of the mode q for different neutrino flows. The quantities $P_{\nu,\text{cdm}}$ are defined so that the right hand sides of eqs. (4.8) and (4.9) read $4\pi k^2/3 \int dq P_{\nu,\text{cdm}}(q)$. The values of τ range from $2.25 k_B T_0$ (bottom line) to $18 k_B T_0$ (top line). On the left panel μ_k is set to 0, on the right panel μ_k is set to 1 and the neutrino mass is set to 0.3 eV. The gray dashed line represents the power spectrum of the cold dark matter displacement.

5 Conclusion

Describing neutrinos as a collection of single-stream fluids is an efficient strategy to infer their impact on the growth of the cosmic structure. Using this approach, one indeed gets a complete set of equations of motion that incorporate all the nonlinear effects of relativistic or non-relativistic particles. In the subhorizon limit, the system takes the form of eq. (2.13), which can be easily handled by a formalism originally developed to depict non-relativistic species.

In this paper, we evaluated the amplitudes of the nonlinear couplings and determined for each flow the scales at which they are expected to impact significantly on the structure growth. For that purpose, we implemented the eikonal approximation into the general equation of motion. We concluded that the impact of large-scale modes on an arbitrary mode are entirely driven by large-scale displacement fields whose expressions are given in eq. (3.6). The comparison between the displacement field associated with each flow of neutrinos and the one associated with cold dark matter makes easy the comparison between the power spectra of the relative displacements between neutrinos and cold dark matter and the power spectrum of the displacement of cold dark matter alone. We found as a preliminary result that couplings involving massive neutrinos (with a 0.3 eV mass) are expected to induce a damping of the perturbation growth in neutrino flows for wave numbers larger than (or of the order of) about 0.2 to 0.5 h/Mpc . A detailed quantitative analysis of the consequences of this phenomenon is yet to be done but those findings confirm the significance of nonlinear couplings in the dynamical evolution of neutrino fluids. This sets the stage for further numerical studies beyond the linear regime.

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A Integral form of the overall equation of motion

For a finite number of flows, the overall equation of motion (2.13) can formally be written in an integral form. It requires the use of the associated Green operator, $g_a^b(z, z_0; \mathbf{k})$, of the linear system. This operator satisfies

$$\Psi_a(z, \mathbf{k}) = g_a^b(z, z_0; \mathbf{k}) \Psi_b(z_0, \mathbf{k}), \quad (\text{A.1})$$

with z and z_0 two arbitrary times. It is besides solution of the differential equation

$$\frac{\partial}{\partial z} g_a^b(z, z_0; \mathbf{k}) + \Omega_a^c(z; \mathbf{k}) g_c^b(z, z_0; \mathbf{k}) = 0 \quad (\text{A.2})$$

with the condition

$$g_a^b(z_0, z_0; \mathbf{k}) = \delta_a^b, \quad (\text{A.3})$$

δ_a^b being the identity matrix. Formally, the Green function is the ensemble of all the independent linear solutions of the system⁵. Denoting $u_a^{(\alpha)}(z, \mathbf{k})$ these solutions, g_a^b reads

$$g_a^b(z, z_0, \mathbf{k}) = \sum_{\alpha} u_a^{(\alpha)}(z, \mathbf{k}) c_{(\alpha)}^b(z_0, \mathbf{k}), \quad (\text{A.4})$$

where the variables $c_{(\alpha)}^b(z_0, \mathbf{k})$ are set so that (A.3) is satisfied.

Studying in detail the Green operator of such a system is beyond the scope of this appendix. Suffice to note here that, unlike the case of a single pressureless flow, the Green operator generally depends on the wave mode k . This dependence is expected to gradually decay over time and to disappear at very late time, when all the flows have become non-relativistic. At this stage, the situation is then identical to the one of a collection of cold dark matter fluids.

As for the standard system of non-relativistic particles, the knowledge of the Green operator of the equation of motion allows to write a formal solution (see [26–28]), which is given by

$$\begin{aligned} \Psi_a(\mathbf{k}, z) = & g_a^b(\mathbf{k}, z, z_0) \Psi_b(\mathbf{k}, z_0) + \\ & + \int_{z_0}^z dz' g_a^b(\mathbf{k}, z, z') \gamma_b^{cd}(\mathbf{k}_1, \mathbf{k}_2) \Psi_c(\mathbf{k}_1, z') \Psi_d(\mathbf{k}_2, z'), \end{aligned} \quad (\text{A.5})$$

with $\Psi_a(\mathbf{k}, z_0)$ the initial conditions. Many of the approaches developed in order to improve upon standard Perturbation Theory rely on an accurate description of the Green functions beyond the linear regime. This is the purpose for instance of RPT and RegPT methods ([28–30]).

References

- [1] J. Lesgourgues, S. Matarrese, M. Pietroni, and A. Riotto, *Non-linear power spectrum including massive neutrinos: the time-RG flow approach*, *J. of Cosmology and Astr. Phys.* **6** (June, 2009) 17, [[arXiv:0901.4550](#)].
- [2] M. Shoji and E. Komatsu, *Massive neutrinos in cosmology: Analytic solutions and fluid approximation*, *Phys. Rev. D* **81** (June, 2010) 123516.

⁵A priori the number of solutions is equal to twice the number of flows considered.

- [3] D. Blas, M. Garny, T. Konstandin, and J. Lesgourgues, *Structure formation with massive neutrinos: going beyond linear theory*, *ArXiv e-prints* (Aug., 2014) [[arXiv:1408.2995](#)].
- [4] S. Saito, M. Takada, and A. Taruya, *Neutrino mass constraint from the Sloan Digital Sky Survey power spectrum of luminous red galaxies and perturbation theory*, *Phys. Rev. D* **83** (Feb., 2011) 043529, [[arXiv:1006.4845](#)].
- [5] S. Riemer-Sørensen, C. Blake, D. Parkinson, T. M. Davis, S. Brough, M. Colless, C. Contreras, W. Couch, S. Croom, D. Croton, M. J. Drinkwater, K. Forster, D. Gilbank, M. Gladders, K. Glazebrook, B. Jelliffe, R. J. Jurek, I.-h. Li, B. Madore, D. C. Martin, K. Pimbblet, G. B. Poole, M. Pracy, R. Sharp, E. Wisnioski, D. Woods, T. K. Wyder, and H. K. C. Yee, *WiggleZ Dark Energy Survey: Cosmological neutrino mass constraint from blue high-redshift galaxies*, *Phys. Rev. D* **85** (Apr., 2012) 081101, [[arXiv:1112.4940](#)].
- [6] B. Audren, J. Lesgourgues, S. Bird, M. G. Haehnelt, and M. Viel, *Neutrino masses and cosmological parameters from a Euclid-like survey: Markov Chain Monte Carlo forecasts including theoretical errors*, *J. of Cosmology and Astr. Phys.* **1** (Jan., 2013) 26, [[arXiv:1210.2194](#)].
- [7] C. Carbone, L. Verde, Y. Wang, and A. Cimatti, *Neutrino constraints from future nearly all-sky spectroscopic galaxy surveys*, *J. of Cosmology and Astr. Phys.* **3** (Mar., 2011) 30, [[arXiv:1012.2868](#)].
- [8] R. Laureijs, J. Amiaux, S. Arduini, J. . Auguères, J. Brinchmann, R. Cole, M. Cropper, C. Dabin, L. Duvet, A. Ealet, and et al., *Euclid Definition Study Report*, *ArXiv e-prints* (Oct., 2011) [[arXiv:1110.3193](#)].
- [9] I. Tereno, C. Schimd, J.-P. Uzan, M. Kilbinger, F. H. Vincent, and L. Fu, *CFHTLS weak-lensing constraints on the neutrino masses*, *Astr. & Astrophys.* **500** (June, 2009) 657–665, [[arXiv:0810.0555](#)].
- [10] C.-P. Ma and E. Bertschinger, *A calculation of the full neutrino phase space in cold + hot dark matter models*, *Astrophys. J.* **429** (July, 1994) 22–28, [[astro-ph/](#)].
- [11] C.-P. Ma and E. Bertschinger, *Cosmological perturbation theory in the synchronous and conformal Newtonian gauges*, *Astrophys. J.* **455** (1995) 7–25, [[astro-ph/9506072](#)].
- [12] J. Lesgourgues and S. Pastor, *Massive neutrinos and cosmology*, *Phys. Rept.* **429** (2006) 307–379, [[astro-ph/0603494](#)].
- [13] M. Pietroni, *Flowing with time: a new approach to non-linear cosmological perturbations*, *J. of Cosmology and Astr. Phys.* **10** (Oct., 2008) 36, [[arXiv:0806.0971](#)].
- [14] F. Führer and Y. Y. Y. Wong, *Higher-order massive neutrino perturbations in large-scale structure*, *ArXiv e-prints* (Dec., 2014) [[arXiv:1412.2764](#)].
- [15] N. Van de Rijdt, *Signatures of the primordial universe in large-scale structure surveys*. PhD thesis, Ecole Polytechnique & Institut de Physique Théorique, CEA Saclay, 2012.
- [16] H. Dupuy and F. Bernardeau, *Describing massive neutrinos in cosmology as a collection of independent flows*, *J. of Cosmology and Astr. Phys.* **1** (Jan., 2014) 30, [[arXiv:1311.5487](#)].
- [17] H. Dupuy and F. Bernardeau, *Cosmological Perturbation Theory for streams of relativistic particles*, *ArXiv e-prints* (Nov., 2014) [[arXiv:1411.0428](#)].
- [18] F. Bernardeau, N. van de Rijdt, and F. Vernizzi, *Resummed propagators in multicomponent cosmic fluids with the eikonal approximation*, *Phys. Rev. D* **85** (Mar., 2012) 063509, [[arXiv:1109.3400](#)].
- [19] F. Bernardeau, S. Colombi, E. Gaztañaga, and R. Scoccimarro, *Large-scale structure of the Universe and cosmological perturbation theory*, *Phys. Rep.* **367** (Sept., 2002) 1–3.

- [20] H. D. I. Abarbanel and C. Itzykson, *Relativistic eikonal expansion*, *Phys. Rev. Lett.* **23** (1969) 53.
- [21] D. Tseliakhovich and C. Hirata, *Relative velocity of dark matter and baryonic fluids and the formation of the first structures*, *Phys. Rev. D* **82** (Oct., 2010) 083520–+, [[arXiv:1005.2416](#)].
- [22] F. Bernardeau, N. Van de Rijt, and F. Vernizzi, *Power spectra in the eikonal approximation with adiabatic and nonadiabatic modes*, *Phys. Rev. D* **87** (Feb., 2013) 043530, [[arXiv:1209.3662](#)].
- [23] M. Crocce and R. Scoccimarro, *Memory of initial conditions in gravitational clustering*, *Phys. Rev. D* **73** (Mar., 2006) 063520–+, [[astro-ph/0509419](#)].
- [24] F. Bernardeau, M. Crocce, and E. Sefusatti, *Multipoint propagators for non-Gaussian initial conditions*, *Phys. Rev. D* **82** (Oct., 2010) 083507, [[arXiv:1006.4656](#)].
- [25] F. Bernardeau, *The evolution of the large-scale structure of the universe: beyond the linear regime*, *ArXiv e-prints* (Nov., 2013) [[arXiv:1311.2724](#)].
- [26] R. Scoccimarro, *Transients from initial conditions: a perturbative analysis*, *Mon. Not. R. Astr. Soc.* **299** (Oct., 1998) 1097–1118, [[astro-ph/](#)].
- [27] R. Scoccimarro, *A New Angle on Gravitational Clustering*, in *The Onset of Nonlinearity in Cosmology* (J. N. Fry, J. R. Buchler, and H. Kandrup, eds.), vol. 927 of *New York Academy Sciences Annals*, pp. 13–+, 2001.
- [28] M. Crocce and R. Scoccimarro, *Renormalized cosmological perturbation theory*, *Phys. Rev. D* **73** (Mar., 2006) 063519, [[astro-ph/](#)].
- [29] F. Bernardeau, A. Taruya, and T. Nishimichi, *Cosmic propagators at two-loop order*, *Phys. Rev. D* **89** (Jan., 2014) 023502.
- [30] A. Taruya, F. Bernardeau, T. Nishimichi, and S. Codis, *Direct and fast calculation of regularized cosmological power spectrum at two-loop order*, *Phys. Rev. D* **86** (Nov., 2012) 103528, [[arXiv:1208.1191](#)].